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An Algorithm for the Normal Forms of Cubic Curves

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§1. Introduction

Normal forms of singularities have been studied by (for instance) Arnold [1, 2, 3]. Bruce and wall [4]. Takahashi. Watanabe and Higuchi [5]. In previous paper [6], we have given a recognition principle for a hypersurface isolated singularity of a certain type. In it. a normal form of homogeneous polynomial was constructed by the monomials of the lowest degree. However. the normal forms constructed by the principle were not unique. So, in this paper. we try to impose a condition to construct a unique normal form of homogeneous polynomial and classify complex projective cubic curves.

We consider it natural that normal form should be easy to write and remember; that is. the normal form should have the fewest monomials. and each monomial should be simple. The normal forms defined in this paper meet the above condition.

§2. A condition for the normal form to be unique

Let $f=\Sigma a_i x^{K_i}$ be a homogeneous polynomial. We give a following order to the monomials of f.

Definition2.1. For the exponents $K_i=k_{i_1},\cdots\cdots,k_{i_n}$ and $K_j=k_{j_1},\cdots\cdots,k_{j_n}$, K_i is grater than K_j if $k_{i_1}>k_{j_1}$ or $k_{i_p}=k_{j_p}$ $(1\leqq p<n)$, $k_{i_{p+1}}>k_{j_{p+1}}$.

Manipulation2.2. We try to make a monomial x^{K_i} vanish by suitable linear transformations except for the moduli. Then if we can make the monomial x^{K_i} vanish (K_i is the minimal number of the exponents) without generating new monomial x^{K_j} ($K_i > K_j$) of f. we do so. Otherwise, we don't use the linear transformations.

Definition2.3. We repeat this manipulation2.2. in turn for i. f is said to be the normal form if the result of these manipulations is equal to f.

§3. Classification of cubic curves

Let P^2 be a 2-dimensional complex projective space with a coordinate $[x, y, z]$ and let f be a

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cubic form in $\mathbf{P}^2$. The cubic form f takes the following form

$a_1x^3+a_2x^2y+a_3x^2z+a_4xy^2+a_5xyz+a_6xz^2+a_7y^3+a_8y^2z+a_9yz^2+a_{10}z^3=0$.

And we may choose coordinates so that

$x^3+a_1x^2y+a_2x^2z+a_3xy^2+a_4xyz+a_5xz^2+y^3+a_6y^2z+a_7yz^2+z^3=0$.

Let Sub. $(x=x', y=y', z=z'+cx', f)$:= [substitution $x=x', y=y', z=z'+cx'$, to equation f] for c is constant so that $c^3+a_5c^2+a_2c+1=0$.

Then $a_1x^2y+a_2x^2z+a_3xy^2+a_4xyz+a_5xz^2+y^3+a_6y^2z+a_7yz^2+z^3=0$.

$a_2\neq 0\to$Step1, $a_2=0$ and $a_1\neq 0\to$step2, $a_2=a_1=0\to$Step3

Step1 : We may choose coordinates so that $a_1x^2y+x^2z+a_2xy^2+a_3xyz+a_4xz^2+y^3+a_5y^2z+a_6yz^2+z^3=0$.

Let Sub. $(x=x', y=y', z=z'-a_1y')$. Then $x^2z+a_1xy^2+a_2xyz+a_3xz^2+a_4y^3+a_5y^2z+a_6yz^2+z^3=0$.

$a_1\neq 0\to$Step4, $a_1=0$ and $a_4\neq 0\to$step5, $a_1=a_4=0\to$Step6.

Step2 : We may choose coordinates so that

$x^2y+a_1xy^2+a_2xyz+a_3xz^2+y^3+a_4y^2z+a_5yz^2+z^3=0$. Let Sub. $(x=x', y=z', z=y')$.

$a_3\neq 0\to$Step4, $a_3=0\to$Step5.

Step3 : We may choose coordinates so that $f_2(y,z)x+f_3(y,z)=0$ where f_i denotes a homogeneous polynomial of degree $i(i=2,3)$. $f_2(y,z)\approx yz\to$Step17, $f_2(y,z)\approx z^2\to$Step18, $f_2(y,z)\equiv 0\to$step22.

Step4 : We may choose coordinates so that $x^2z+xy^2+a_1xyz+a_2xz^2+a_3y^3+a_4y^2z+a_5yz^2+z^3=0$. Let Sub. $(x=x'-a_1y'/2-a_2z'/2, y=y', z=z')$. Then $x^2z+xy^2+a_1y^3+a_2y^2z+a_3yz^2+a_4z^3=0$.

$a_4\neq 0\to$Step7, $a_4=0$ and $a_3\neq 0\to$Step8, $a_4=a_3=0\to$Step9.

Step5 : We may choose coordinates so that $x^2z+a_1xyz+a_2xz^2+y^3+a_3y^2z+a_4yz^2+z^3=0$.

Let Sub. $(x=x'-a_1y'/2-a_2z', y=y', z=z')$. Then $x^2z+y^3+a_1y^2z+a_2yz^2+a_3z^3=0$. Go to Step10.

Step6 : We may choose coordinates so that $x^2z+a_1xyz+a_2xz^2+a_3y^2z+a_4yz^2+z^3=0$.

Let Sub. $(x=x'-a_1y'/2-a_2z'/2, y=y', z=z')$. Then $x^2z+a_1y^2z+a_2yz^2+a_3z^3=0$. $a_1\neq 0\to$Step13, $a_1=0\to$Step14.

Step7 : We may choose coordinates so that $x^2z+xy^2+a_1y^3+a_2y^2z+a_3yz^2+z^3=0$.

Let Sub. $(x=c_1x'+c_2y'-z', y=\alpha x'+y', z=x')$ for α is a solution of the following algebraic equation :

$\alpha^3-3a_1\alpha^2-2a_2\alpha-a_3-\sqrt{3\alpha^2-6a_1\alpha-2a_2-g_1(\alpha)}\,g_1(\alpha)/\sqrt{2}=0$,

where $g_1(\alpha)=\sqrt{\alpha^4-4a_1\alpha^3-4a_2\alpha^2-4a_3\alpha-4}$, and $c_1=(\sqrt{\alpha^4-4a_1\alpha^3-4a_2\alpha^2-4a_3\alpha-4}-\alpha^2)/2$ $(=(g_1(\alpha)-\alpha^2)/2)$, $c_2=(\sqrt{-g_1(\alpha)+3\alpha^2-6a_1\alpha-2a_2}-\sqrt{2}\alpha)/\sqrt{2}$.

Then $a_1x^2z+a_2xyz+xz^2+a_3y^3-y^2z=0$. $a_1\neq 0$ and $a_3\neq 0\to$Step5, $a_1\neq 0$ and $a_3=0\to$Step6, $a_1=0\to$Step3.

Here $a_1=-g_1(\alpha), a_2=-2\sqrt{-g_1(\alpha)+3\alpha^2-6a_1\alpha-2a_2}/\sqrt{2}$,

$a_3=\sqrt{-g_1(\alpha)+3\alpha^2-6a_1\alpha-2a_2}-\sqrt{2}\alpha+a_1\sqrt{2})/\sqrt{2}$.

$$\begin{vmatrix} c_1 & c_2 & -1 \\ \alpha & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1.$$

Step8 : We may choose coordinates so that $x^2z+xy^2+a_1y^3+a_2y^2z+yz^2=0$.

Let Sub. $(x=c_1x'+c_2y'-z',\ y=\alpha x'+y',\ z=x')$ for α is a solution of the following algebraic equation :

$\alpha^3-3a_1\alpha^2-2a_2\alpha-1-\beta\sqrt{3\alpha^2-6a_1\alpha-2a_2-\beta g_1(\alpha)}\,g_1(\alpha)/\sqrt{2}=0$,

where $g_1(\alpha)=\sqrt{\alpha^3-4a_1\alpha^2-4a_2\alpha-4}$, $\beta=\sqrt{\alpha}$, and $c_1=(\beta g_1(\alpha)-\alpha^2)/2$,

$c_2=(\sqrt{-\beta g_1(\alpha)+3\alpha^2-6a_1\alpha-2a_2}-\sqrt{2})\sqrt{2}$.

Then $a_1x^2z+a_2xyz+xz^2+a_3y^3-y^2z=0$.

Here $a_1=-\beta g_1(\alpha)$, $a_2=-2\sqrt{-\beta g_1(\alpha)+3\alpha^2-6a_1\alpha-2a_2}/\sqrt{2}$,

$a_3=\sqrt{-\beta g_1(\alpha)+3\alpha^2-6a_1\alpha-2a_2}-\sqrt{2}+\sqrt{2}a_1)/\sqrt{2}$. $\quad \begin{vmatrix} c_1 & c_2 & -1 \\ \alpha & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1.$

$a_1\neq 0$ and $a_3\neq 0\to$Step5. $a_1\neq 0$ and $a_3=0\to$Step6. $a_1=0\to$Step3.

Step9 : We may choose coordinates so that $x^2z+xy^2+a_1y^3+a_2y^2z=0$.

Let Sub. $(x=z',\ y=y',\ z=x')$. Go to Step3.

Step10 : We may choose coordinates so that $x^2z+y^3+a_1y^2z+a_2yz^2+a_3z^3=0$.

Let Sub. $(x=x',\ y=y'+cz',\ z=z')$ for c is a constant so that $3c^2+2a_1c-a_2=0$.

Then $x^2z+y^3+a_1y^2z+a_2z^3=0$. $a_2\neq 0\to$ Step11. $a_2=0\to$Step12.

Step11 : We may choose coordinates so that $x^2z+y^3+ay^2z+z^3=0$. $4a^3+27\neq 0$. Non-singular Elliptic Curve.

Step12 : We may choose coordinates so that $x^2z+y^3+y^2z=0$.

Let Sub. $(x=z',\ y=y',\ z=x')$. Go to Step3.

Step13 : We may choose coordinates so that $x^2z+y^2z+a_1yz^2+a_2z^3=0$.

Let Sub. $(x=x',\ y=y'-a_1z'/2,\ z=z')$. Then $x^2z+y^2z+az^3=0$. $a\neq 0\to$Step15. $a=0\to$Step16.

Step14 : We may choose coordinates so that $x^2z+a_1yz^2+a_2z^3=0$.

Let Sub. $(x=y',\ y=x',\ z=z')$. Go to Step3.

Step15 : We may choose coordinates so that $x^2z+y^2z+z^3=0$.

Let Sub. $(x=x',\ y=y'+\sqrt{-1}x',\ z=z')$. Then $y^2z+2\sqrt{-1}xyz+z^3=0$. Go to Step3.

Step16 : we may choose coordinates so that $x^2z+y^2z=0$.

Let Sub. $(x=z',\ y=y',\ z=x')$. Go to Step3.

Step17 : We may choose coordinates so that $xyz+a_1y^3+a_2y^2z+a_3yz^2+a_4z^3=0$.
Let Sub. $(x=x'-a_2y'-a_3z',\ y=y',\ z=z')$. Then $xyz+a_1y^3+a_2z^3=0$. $a_1\neq 0$ and $a_2\neq 0$ →Step23. $\{a_1\neq 0$ and $a_2=0\}$ or $\{a_1\neq 0$ and $a_2=0\}$ →Step24. $a_1=a_2=0$→Step25.

Step18 : We may choose coordinates so that $xz^2+a_1y^3+a_2y^2z+a_3yz^2+a_4z^3=0$. $a_1=0$ →Step 19. $a_1=0$ and $a_2\neq 0$→Step20. $a_1=a_2=0$→Step21.

Step19 : We may choose coordinates so that $xz^2+y^3+a_1y^2z+a_2yz^2+a_3z^3=0$.
Let Sub. $(x=x'+(-3a_2+a_1^2)y'/3+(-27a_3+9a_1a_2-2a_1^3)z',\ y=y'-a_1z'/3,\ z=z'$
Then $xz^2+y^3=0$. Cuspidal Curve.

Step20 : We may choose coordinates so that $xz^2+y^2z+a_1yz^2+a_2z^3=0$.
Let Sub. $(x=x'-a_1y'-a_2z',\ y=y',\ z=z')$. Then $xz^2+y^2z=0$. Conic and Tangent.

Step21 : We may choose coordinates so that $xz^2+a_1yz^2+a_2z^3=0$.
Let Sub. $(x=x'-a_1y'-a_2z',\ y=y',\ z=z')$. Then xz^2. Go to Step26.

Step22 : $f_3(y, z)\approx yz^2$→Multiple and Single Lines.
$f_3(y, z)\approx z^3$→Triple Line.
$f_3(y, z)\approx y^2z+z^3$→Three Concurrent Lines.

Step23 : We may choose coordinates so that $xyz+y^3+z^3=0$. Nodal Curve.

Step24 : We may choose coordinates so that $xyz+z^3=0$. Conic and Chord.

Step25 : We may choose coordinates so that $xyz=0$. Three General Lines.

Step26 : We may choose coordinates so that $xz^2=0$.
Let Sub. $(x=y',\ y=x',\ z=z')$.
Then $yz^2=0$. Multiple and Single Lines.

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F1:=X**2*Z$
F2:=X*(Y**2+A1*Y*Z+A2*Z**2)$
F3:=A3*Y**3+A4*Y**2*Z+A5*Y*Z**2+A6*Z**3$
F:=F1+F2+F3;
C3:=1$
C5:=1$
C7:=1$
C6:=0$
C8:=0$
C9:=0$
F1:=SUB(X=C1*A+C2*B+C3*C,Y=C4*A+C5*B+C6*C,Z=C7*A+C8*B+C9*C,F)$
CLEAR F$
MATRIX M$
M:=MAT((C1,C2,C3),(C4,C5,C6),(C7,C8,C9));
G1:=DF(F1,A,3)/(3*2);
G2:=DF(F1,A,2,B)/2;
G3:=DF(F1,A,2,C)/2;
G4:=DF(F1,A,B,2)/2;
G5:=DF(F1,A,B,C);
G6:=DF(F1,A,C,2)/2;
G7:=DF(F1,B,3)/(3*2);
G8:=DF(F1,B,2,C)/2;
G9:=DF(F1,B,C,2)/2;
G10:=DF(F1,C,3)/(3*2);
SOLVE(G4,C2);
C2:=SOLN(1,1)$
G1$
G2$
G4$
H1:=-A1**2*C4-A1*A2-3*A1*C4**2-2*A2*C4$
H2:=6*A3*C4**2+4*A4*C4+2*A5-2*C4**3$
H3:=(H1+H2)/2$
CLEAR H1,H2$
H4:=G2-H3$
H5:=H4**2-H3**2$
CLEAR H3,H4$
G1;
SOLVE(G1,C1);
C1:=SOLN(1,1);
G1;
H5;
J1:=A1**3*C4**3+3*A1**2*A2*C4**2-2*A1**2*A3*C4**3$
J2:=-2*A1**2*A6+3*A1**2*C4**4+3*A1*A2**2*C4$
J3:=-6*A1*A2*A3*C4**2+2*A1*A2*A5+6*A1*A2*C4**3$
J4:=-6*A1*A3*C4**4-4*A1*A4*C4**3-6*A1*A5*C4**2$
J5:=-12*A1*A6*C4+3*A1*C4**5+A2**3-6*A2**2*A3*C4$
J6:=-2*A2**2*A4+3*A2**2*C4**2-4*A2*A3*C4**3-4*A2*A6$
J7:=3*A2*C4**4+6*A3**2*C4**4+8*A3*A4*C4**3+12*A3*A5*C4**2$
J8:=24*A3*A6*C4-6*A3*C4**5+8*A4*A6-6*A4*C4**4-2*A5**2$
J9:=-8*A5*C4**3-12*A6*C4**2+C4**6$
J10:=(J1+J2+J3+J4+J5+J6+J7+J8+J9)/2$
CLEAR J1,J2,J3,J4,J5,J6,J7,J8,J9$
J11:=H5-J10$
J12:=J11**2-J10**2$
CLEAR J10,J11$
ORDER C4$
J12$
CLEAR G1,G2,G3,G4,G5,G6,G7,G8,G9,G10$
G9:=DF(J12,C4,9)/(9*8*7*6*5*4*3*2)$
G8:=SUB(C4=0,DF(J12,C4,8)/(8*7*6*5*4*3*2))$
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G7:=SUB(C4=0,DF(J12,C4,7)/(7*6*5*4*3*2))$
G6:=SUB(C4=0,DF(J12,C4,6)/(6*5*4*3*2))$
G5:=SUB(C4=0,DF(J12,C4,5)/(5*4*3*2))$
G4:=SUB(C4=0,DF(J12,C4,4)/(4*3*2))$
G3:=SUB(C4=0,DF(J12,C4,3)/(3*2))$
G2:=SUB(C4=0,DF(J12,C4,2)/2)$
G1:=SUB(C4=0,DF(J12,C4))$
G0:=SUB(C4=0,J12)$
G9;
SOLVE(G9,A5);
A5:=SOLN(1,1);
G9;
G8;
SOLVE(G8,A6);
A6:=SOLN(1,1);
G8;
G7;
G6;
G5;
G4;
G3;
G2;
G1;
G0;
COMMENT: THEREFORE, THERE EXIST THE SOLUTIONS. ;
DET(M);
CLEAR J12,H5;
CLEAR C1,C2,C3,C4,C5,C6,C7,C8,C9;
CLEAR F1,M,G0,G1,G2,G3,G4,G5,G6,G7,G8,G9;
COMMENT: STEP4 IS COMPLETED. ;
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